Perfectoid unitary Shimura varieties and the overconvergent Eichler-Shimura map

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Overview

1. Introduction
2. Perfectoid automorphic forms
3. Overconvergent Eichler-Shimura map
4. Further development

Long-term goals:
Use π_{HT} to study p-adic automorphic forms and related topics.
starting points:

- Use π_{HT} and $\mathcal{F}\ell$ (Flag variety) to study geometry of perfectoid Shimura varieties.
- Relate perfectoid Shimura varieties and finite level (usual) Shimura varieties.
- Develop p-adic analogue of complex theory.

First, let us recall the picture about modular forms in complex setting.
Let $\mathbb{H}$ denote the upper half-plane, and $\Gamma = \Gamma_1(N)$ denote the arithmetic subgroup, and $X = X_1(N)$ denote the modular curve with level Γ. For simplicty we ignore issues about boundary.
Through the complex uniformization $\mathbb{H} \longrightarrow X$, the modular bundle ω^k is canonically trivialized, and we can identify weight k modular forms with analytic functions on $\mathbb{H}$ satisfying

$$f(\frac{az+b}{cz+d}) = (cz+d)^k f(z), \ \textit{for } \gamma \in \Gamma.$$

Here $j(\gamma, z) = (cz+d)^k$ is automorphy factor.

On the other hand, the Eichler-Shimura map is an edge map for the dual BGG spectral sequence. This theory is about comparison of Betti (etale) and coherent cohomology of Shimura varieties. It can seen as a generalization of Hodge-Derham spectral sequence.
GL_2:
Over X, we have the resolution

$$0 \longrightarrow Std^k \longrightarrow \omega^{-k} \longrightarrow \omega^{k+2} \longrightarrow 0.$$

It induces the following exact sequence:

$$0 \longrightarrow H^0(X, \omega^{k+2}) \longrightarrow H^1(X, Std^k) \longrightarrow H^1(X, \omega^{-k}) \longrightarrow 0.$$

Introduction

What about p-adic setting?

• Faltings: classical weight GL_2(...)

New feature in p-adic setting:

We can do ***p*-adic interpolations**.

Overconvergent forms:

• Andreatta-Iovita-Pilloni: GSp_{2g} (many generalizations);

Overconvergent Eichler-Shimura:

• Andretta-Iovita-Stevens: GL_2 (...)

Introduction

New ideas via perfectoid methods:

- Chojecki-Hansen-Johansson: GL_2 (compact);
- Birkbeck-Heuer-Williams: Hilbert case;
- Diao-Giovanni-Wu: GSp_{2g};

Today I will present my work (in progress) about compact $GU(n,1)$ setting.
(later I will also mention another sytle p-adic Eichler-Shimura theory by Pan Lve, Camargo...)

π_∞ is very different from π_{HT}.
We will make the following p-adic analogue:
complex analytic forms v.s overconvergent forms;

Introduction

Key steps:

(1) Define w-ordinary locus through π_{HT};
(2) Define perfectoid automorphic forms;
(3) Comparison with AIP style overconvergent forms;
(4) Define overconvergent cohomology;
(5) Define overconvergent Eichler-Shimura map;
(6) Study the resulting map between coherent sheaves on the eigenvariety;
(7) Apply the Bellaiche's method to study ramification locus of the weight map.

Perfectoid Shimura

Let $E/\mathbb{Q}$ be an imaginary quadratic field that p splits, and we fix a splitting prime over p for E.

Let B/E be a central divison algebra of dimension $(n+1)^2$ that splits at p and some other technical conditions so that the resulting PEL type Shimura variety for $G = GU(n,1)$ is compact.

In particular $G_{\mathbb{Q}_p}$ splits as $GL(n+1, \mathbb{Q}_p) \times GL(1, \mathbb{Q}_p)$. For simplicity we will ignore the $GL(1, \mathbb{Q}_p)$ factor in this talk and write Levi subgroup M $(\cong GL(n) \times GL(1))$. And we will consider weight in the form $(k_1, ..., k_n, 0)$.

Let $K = K_p K^p$ denote the neat level subgroup. We will mainly work with K_p being Iwahori subgroup IW.

Indeed, unlike GL_2 case, higher rank there are some notation issues, for example Diao-Rosso-Wu worked with strict Iwahori subgroup Iw^+. Here we omit such issue.

Perfectoid Shimura

For an abelian variety A, we have the following exact sequence:

$$0 \longrightarrow Lie(A)(1) \longrightarrow T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{C}_p \longrightarrow \omega_{\check{A}} \longrightarrow 0.$$

Roughly speaking, infinite level (at p) Shimura variety parametrizes A with trivalization $\mathbb{Z}_p^{2g} \cong T_p(A)$ (plus other information). Then the Hodge-Tate period map π_{HT} sends such a point x to $Lie(A)$, which is a point on the flag variety $\mathcal{F}\ell$.
In our situation, we have a canonical decomposition

$$A[p^\infty] = H \oplus H_1,$$

where H is n-dimensional p-divisible group with height $n+1$, H_1 is dual to H. Then we can replace A by H. The resulting flag variety is

$$\mathcal{F}\ell = P \backslash G = Gr(n, n+1) \cong P^n.$$

w-ordinary locus

Fix a basis $\{e_1, ..., e_{n+1}\}$ for $V_{\mathbb{Z}_p}$.
We will restrict to the open cell $\mathcal{F}\ell^*$that parametrizes n-dimensional subspace W such that $e_{n+1} \notin W$.
We will write the coordinate for such point x by

$$[I_n \ Z]$$

here $Z = Z(x)$ is an $n \times 1$ matrix.
For each $w \in \mathbb{Q}_{>0}$, we define the following open adic subspace

$$\mathcal{F}\ell_w := \{x \in \mathcal{F}\ell^* : max_i inf_{h \in \mathbb{Z}_p}\{|Z_i(x) - h|\} \leq p^{-w}\}.$$

It is stable under the right multiplication of Iw.

w-ordinary locus

The key fact is that A is ordinary iff $Lie(A)$ is $\mathbb{Q}_p$-rational. Thus we can think of $\mathcal{F}\ell_w$ as w-ordinary locus on the flag variety; When w goes to ∞, it becomes more and more "ordinary".
Pullback to X_∞, we get the open subspace $\mathcal{X}_{\infty,w}$. Through the projection

$$q : X_\infty \longrightarrow X_{Iw},$$

we define $X_{Iw,w}$ to be $q(X_{\infty,w})$. In fact we also have $X_{\infty,w} = q^{-1}(X_{Iw,w})$. In this way we define the w-ordinary locus on the Shimura variety.

On the other hand, Let $W_{\mathcal{F}\ell}$ denote the universal (tautological) n-dimensional bundle over $\mathcal{F}\ell$. And let $\widehat{W_{\mathcal{F}\ell}}$ denote the dual bundle.
The dual bundle has n-canonical sections $s_1, ..., s_n$. Its transform rule under Iw-action is related with automorphy factor.
The intuition is that we have the following fact:
• Let ω denote the universal Hodge bundle by H over X_{Iw}. Then

$$q^*(w) = \pi_{HT}^*(\widehat{W_{\mathcal{F}\ell}})$$

Further we define the coordinate on $X_{\infty,w}$ by

$$\widetilde{Z} = \pi_{HT}^*(Z).$$

Let $\mathcal{W}$ denote the weight space (rigid variety) over $\mathbb{Q}_p$ defined by $(\mathbb{Z}_p^*)^n$.
A small $\mathbb{Z}_p$-algebra R is a p-torsion free reduced ring which is also a finite $\mathbb{Z}_p[[T_1, ..., T_d]]$-algebra for some d. It has a canonical adic profinite topology and is also completed under p-adic topology.
A small weight is a pair (R_U, k_U) where R_u is a small Z_p-algebra and $k_U : (\mathbb{Z}_p^*)^n \to R_U^*$ is a continuous group map with $k_U((1+p)\mathbb{I}) - 1$ is topological nilpotent in R_U (under p-adic topology).
Then there is a map

$$Spa(R_u, R_u)^{rig} \longrightarrow \mathcal{W}.$$

Remark

For simplicity, we may omit U. And indeed to define weight k perfectoid forms, we can also use affinoid weight directly. But we need small weight to study overconvergent cohomology. It involves certain integral filtration. The reason (very roughly) is similar to definition of etale cohomology. Even if we only care about rational version, we need to first define integral version and then invert p.

It is well known that weight (R, k) is r_k-analytic for large enough r_k. Now let $w \in Q_{>0}$ with $w > 1 + r_k$. The group Iw acts on w-analytic induction $C_k^{w-an}(Iw_M, BR)$ from left, where BR is a Banach algebra over R. Indeed there is another kind of left action involving the notation issue.

Now we can define the **sheaf of w-overconvergent w-analytic automorphic forms of Iwahori level and weight k** as following:
The sheaf ω_w^k is a subsheaf of $q_* C_k^{w-an}(Iw_M, \mathcal{O}_{X_{\infty,w}} \widehat{\otimes} R)$ such that its section over any affinoid open subset $\mathcal{V}$ corresponds to ∞-level section f for $\mathcal{V}_\infty$ satisfying

$$g^*(f) = \rho(A + \widetilde{Z}C)^{-1} f, \ \ for\ g \in Iw_G.$$

The space of weight k and Iwahori level overconvergent automorphic forms is the direct limit

$$\mathcal{S}_{Iw}^k := lim\ H^0(X_{Iw,w}, \omega^k)$$

AIP method

Before the work of Scholze (about π_{HT}), there are already a construction of overconvergent automorphic forms by the method of AIP.
While AIP did the Siegel case, here I only list two other results about unitary Shimura variety.
Xu Shen did the case $GU(1, n)$ (compact) and Riccardo Brasca did the general case $GU(n_1, n_2)$ (and allow non-compact case). Their work is about totally real field indeed.

AIP method

The AIP method is through the construction of AIP torsor. Roughly speaking, it is similar to Igusa torsor.
And they are using truncated Hodge height v to measure the distance to being ordinary.
We will denote $X_{Iw}(v)$ for the locus with Hodge height not larger than v. Their resulting v-overconvergent w-analytic sheaf is denoted by $\omega_v^{k,AIP}$. Here the range of w is related with v.

comparison two construction

This part is technical and involves many computations, here we only mention two key ideas:

(1) compare the locus $X_{Iw,w}$ and $X_{Iw}(v)$;

(2) compare different torsor and then compare sheaf ω_w^k and $\omega_v^{k,AIP}$.

Theorem

For any weight k, the resulting space of weight k overconvergent forms by perfectoid method is the same as the space of weight k overconvergent forms by AIP method.

Overconvergent cohomology group

Now we turn to Betti (etale) side. Our construction of overconvergent cohomology is a generalization of modular symbol. It is similar to Hansen's construction in his paper about eigenvariety. But the new insight is to use **pro-etale topology**.
Fix weight (R, k). We can consider the space of r-analytic R-valued function on $N^{op}(p\mathbb{Z}_p)$ and identify it with another function space related with Levi M, $A_k^r(\widetilde{M}, R)$.
We will use the dual space (distribution), $D_k^r(\widetilde{M}, R)$. It has certain nice filtration.
The Iw_G action on $D_k^r(\widetilde{M}, R)$ produces an etale sheaves $\mathcal{D}_k^r$.
Let $v : X_{pro-et} \longrightarrow X_{et}$ denote the natural map.
Pullback the filtration through v and take inverse limit, we get another sheaf $\mathcal{O}\mathcal{D}_k^r$.

Eichler-Shimura

We have the following identification:

$$\mathcal{O}\mathcal{D}_k^r \cong (D_k^r(\widetilde{M}, R)\widehat{\otimes} q_* \widehat{\mathcal{O}_{X_{Iw,proet}}})^{Iw_G}.$$

In some sense it is closer to our description of perfectoid forms now.
On the other hand, we can similarly define sheaf on the proetale site $\widehat{\omega_w^k}$. It is also certain twisted Iw_G invariant with the following important facts:

- projection formula

$$\omega_w^k \otimes_{\mathcal{O}_{\mathcal{X}\,Iw,w,et}} R^i v_* \widehat{\mathcal{O}}_{X_{Iw,w,proet}} \cong R^i v_* \widehat{\omega}_w^k.$$

- computation

$$R^i v_* \widehat{\mathcal{O}}_{X_{Iw,w,proet}} \cong \Omega^i_{X_{Iw,w,et}}(-i).$$

Eichler-Shimura

Now we can definte Eichler-Shimura between sheaf. Roughly speaking, we are using Through an anlogue of highest weight vector $e_{hwt,k}$ for M (indeed GL_n). Taking Iw_G-invariant, and we get the desired comparison map:

$$H^n_{proet}(X, \mathcal{O}\mathcal{D}^r_k) \xrightarrow{res} H^n_{proet}(X_w, \mathcal{O}\mathcal{D}^r_k)$$

$$\xrightarrow{ES} H^n_{proet}(X_w, \widehat{\omega}^k_w)$$

$$\xrightarrow{\cong} H^0(X_w, \omega^k_w \otimes \Omega^n)(-n).$$

Use Kodaira-Spence, the dualing sheaf Ω^n can also be expressed as an automorphic bundle.

Eichler-Shimura

Our construction is canonical, Hecke equivariant and functorial in weight. In particular, we can glue them along the eigenvariety $\mathcal{E}$. Therefore we get the second main theorem:

Theorem

There is a canonical map ES from $\mathcal{V}$ to $\mathcal{M}^+$ over $\mathcal{E}$.

Eicher-Shimura

We can further study the map ES:
(1) Over classical weight, it should be compatible with method by Faltings comparison.
(2) Generically it should be surjection.
(3) Generically (plus regular condition) it splits.
(4) The Kernerl of ES will have a filtration related with higher Coleman theory.
Very Recently, Diao-Rosso-Wu (2025) established the final property for GSp_4.
Such refined theory will have application to eigenvariety (obtain certain etale points for weight map) and p-adic L-functions.

comparison with other p-adic Eichler-Shimura

There is another kind of p-adic Eichler-Shimura theory. It is influenced by Pan Lve's idea, and developed by Camargo (in his thesis), and BCGP (2025).

Although both use the π_{HT} as the starting point, their method and application are different:

(1) They are using locally analytic part of completed cohomology;

(2) Their weight is about Lie algebra.

Further development

Many things to do:
(1) Change from $\mathbb{Q}$ to totally real number field F;
(2) General signaure $GU(n_1, n_2)$. This involves duality, the notation and computation are more complicated.
(3) Non-compact unitary Shimura variety.
(4) Inert p, this is a **new** challenge. The relation between π_{HT} with μ-ordinary locus is hard at present.

Summary

I hope the new framework by π_{HT} will through lights on the topic about p-adic automorphic forms.

Thank you!