Bruhat-Tits building and p-adic period domain (joint work with Xu Shen)

Ruishen Zhao
Morningside Center of Mathematics

2025.05

Overview

Introduction

Classical level raising theory is about modulo p congruence between cuspidal eigenforms. For example, Ribet (in [?]) proved the following theorem:

Theorem

Let $f \in S_2(\Gamma_0(N))$ be a cuspdial eigenform and suppose it is not congruent to an Eisenstein series after modulo p. If there exists another prime l not dividing Np, and f satisfies the following condition

$$a_l(f)^2 \equiv (l+1)^2 (mod\ p),$$

then there exists a cuspidal eigenform $g \in S_2(\Gamma_0(Nl))$ which is new at l and it is congruent to f after modulo p.

Here two Hecke eigenforms are congruent modulo p is their associated Hecke eigenvalue are congruent modulo p.

Introduction

The level raising condition can be reinterpreted via Satake parameters. We call a Satake parameter *degenerate* if the corresponding unramified principal series is reducible. For $GL(2, \mathbb{Q}_l)$, it degenerates if and only if the ratio is l or l^{-1}. Then the level raising condition is equivalent to require that the Satake parameter of $\Pi_{f,l}$ is congruent modulo p to a degenerate Satake parameter.
Roughly speaking, level raising theory is to upgrade the local congruence into global congruence.

Introduction

There are many applications of generalizations about level raising theory. For example, Andrew Wiles used level raising results in his famous paper [**?**] about modularity of elliptic curves. In [**?**] Taylor generalized Ribet's result to definite quaternion algebras. In [**?**], Joel Bellaiche and Phillippe Graftieaux studied level raising congruences for definite $U(3)$.
On the other hand, James Newton developed p-adic analogue of level raising results for definite quaternion algebras (see [**?**]). It is about intersection points on the eigenvariety.
In this talk, we will generalize his results to definite $U(3)$ setting.

Local picture

As motivation, we first illustrate a local picture. We will relate the degenerate Satake parameters to intersection points in the moduli space of the tame L-parameters.
Let $G = GL(n, \mathbb{Q}_l)$, its dual group is GL_n with a trivial Galois action, and let gl_n denote the lie algebra. Following Hellman, we introduce the moduli space $X_{\widehat{G}}$. Let C denote an algebraically closed field with characteristic zero. The space $X_{\widehat{G}}$ represents the functor

$$R \longrightarrow \{(\phi, N) \in (GL_n \times gl_n)(R) | Ad(\phi)(N) = l.N\}$$

on the category of C-algebras.
Its irreducible component is in bijection with the set of GL_n-orbits on the nilpotent cone of gl_n. For such an orbit $[N]$, let $X_{\widehat{G}}^{[N]}$ denote the corresponding irreducible component. The unramified component ($[n] = 0$) is also denoted by $X_{\widehat{G}}^{un}$.

Local picture

For each Satake parameter s, we can associate a point $(s, 0)$ on $X_{\widehat{G}}^{un}$. And if s degenerates, this point will lie at the intersection of different irreducible components inside $X_{\widehat{G}}$.

Proposition

If the Satake parameter s is degenerate, then the corresponding point $(s, 0)$ inside $X_{\widehat{G}}^{un}$ will lie at some other irreducible components.

Local picture

Proof.

Let $I(s)$ denote the corresponding (reducible) unramified principal series. Suppose that its Jordan Holder factors are $\{\pi_j | 0 \leq j \leq m\}$ and let π_0 denote the unramified irreducible representation.

According to the local Langlands correspondence, each π_j corresponds to a GL_n orbit (s_j, N_j) within $X_{\widehat{G}}$.. Due to the parabolic induction functionality, each s_j is conjugated to s_0. So we can assume they are equal. For each positive integer j, N_j is nonzero. Observe that we can rescale it via conjugation, there exists a group map $\rho : GL_1 \longrightarrow GL_n$ such that $Ad(\rho(t))(N_j) = t.N_j$. Then the locus $\{(s_j, tN_j) | t \neq 0\}$ lies in the irreducible component $X_{\widehat{G}}^{[N_j]}$. The point $(s, 0)$ lies in the closure of this locus, therefore it also lies in $X_{\widehat{G}}^{[N_j]}$. $\square$

Local picture

However, this proposition may not hold for other groups. The reason is that the unramified L-packet may have more than one element. In fact, the group $SL(2, \mathbb{Q}_l)$ already shows this new feature.

This group G has two conjugacy classes of maximal open compact subgroups (its Bruhat-Tits tree has two types of vertices). One is represented by $K_0 = SL(2, \mathbb{Z}_l)$, and the other is represented by $K_1 = diag(l, 1)K_0 diag(l^{-1}, 1)$. In addition, there exists a Satake parameter s such that $I(s)$ is reducible with two Jordan-Holder factors π_0 and π_1, each π_j is unramified with respect to K_j. And the point $(s, 0)$ is not an intersection point. For the unramified unitary group $U(3, \mathbb{Q}_l)$, such thing happens similarly. In particular, this suggests that we should consider both conjugacy classes together to study intersection points.

Abelian Ihara lemma

Now we turn to the global setting.
The basic strategy (by Ribet, Taylor etc) goes as follow:
Let $L = H^1_{et}(X_0(N), \mathbb{Z}_p)$ and $M = H^1_{et}(X_0(Nl), \mathbb{Z}_p)$. The two degeneracy maps from $X_0(Nl)$ to $X_0(N)$ induces the level raising (lowering) map

$$i : L^2 \longrightarrow M; \ i^+ : M \longrightarrow L^2.$$

Roughly speaking, the image $im(i)$ is related with *old forms* and the kernel $ker(i^+)$ is related with *new forms*. Through natural pairings on L and M, i^+ is the adjoint map for i^+ and we can further relate the quotient $L^2/i^+i(L^2)$ with $ker(i^+)$. In particular, thinking them as Hecke modules, for those maximal ideal β of the Hecke algebra, which lies in the support of $L^2/i^+i(L^2)$ but doesn't lie in the support of $ker(i)$, it will provide a congruence between old forms and new forms. The **level raising condition** is related with the determinant of i^+i and the **Ihara lemma** is about the support of $ker(i)$.

Abelian Ihara lemma

The p-adic story goes in a similar way. But in p-adic setting, the related modules are infinite dimensional and not self-dual. Newton introduced the dual module of p-adic forms and applied the natural pairing to get similar duality arguments.
Now we introduce basic notations for p-adic forms on the definite unitary groups.
Let E/Q be an imaginary quadratic field. Consider an n-dimensional ($n > 1$) Hermitian space over E and let G denote its unitary group. We assume that G is definite, i.e. $G(\mathbb{R})$ is compact. Fix a prime p that splits in E, then we can identify $G(\mathbb{Q}_p)$ with $GL(n, \mathbb{Q}_p)$.
Let T denote the diagonal torus, B denote the usual Borel subgroup corresponding to upper triangular matrices, N denote the unipotent part and $\bar{N}$ denote the opposite unipotent group. Let Iw_p denote the Iwahori subgroup, i.e. the subgroup of $GL(n, \mathbb{Z}_p)$ that becomes $B(\mathbb{F}_p)$ after modulo p.

notations

Let $\mathcal{U}$ denote an open compact (*level*) subgroup of $G(\mathbb{A}_f)$ with $\mathcal{U} = \mathcal{U}_p \times \mathcal{U}^p$. We call $\mathcal{U}_p$ the *wild level* and assume it is Iw_p. The subgroup $\mathcal{U}^p$ is called *tame level*. Then for any commutative ring R and any **right** Iw_p-module A over R, we can define an R-module $\mathcal{F}(\mathcal{U}, A)$ in the following way:

$$\mathcal{F}(\mathcal{U}, A) = \{f : G(\mathbb{Q})\backslash G(\mathbb{A}_f) \longrightarrow A,\ f(gu) = f(g)u_p \textit{ for all } u \in \mathcal{U}\}.$$

For any function $f : G(\mathbb{A}_f) \longrightarrow A$ and $x \in G(\mathbb{A}_f)$ with $x_p \in \mathcal{U}_p$, we define a new function $f|x : G(\mathbb{A}_f) \longrightarrow A$ by

$$(f|x)(g) = f(gx^{-1})x_p.$$

Then we can also write the above module as

$$\mathcal{F}(U, A) = \{f : G(\mathbb{Q})\backslash G(\mathbb{A}_f) \longrightarrow A,\ f|u = f \textit{ for all } u \in \mathcal{U}\}.$$

By generalized finiteness of class groups, the double coset $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/\mathcal{U}$ is finite. Fix a set of representatives $\{x_i | 1 \leq i \leq h\}$, we have the following isomorphism

$$\mathcal{F}(U, A) \longrightarrow \bigoplus_{i=1}^{h} A^{x_i^{-1} G(\mathbb{Q}) x_i \cap \mathcal{U}},$$

$$f \mapsto (f(x_1), ..., f(x_h)).$$

Moreover, each $x_i^{-1} G(\mathbb{Q}) x_i \cap \mathcal{U}$ is a finite group, and it is trivial if the tame level $\mathcal{U}^p$ is small enough. From now on, we assume that the tame level $\mathcal{U}^p$ is small enough (*neat*). Then $\mathcal{F}(\mathcal{U}, A) \cong A^h$. Although in fact such neat assumption can be removed, it will simplify some notations (such as the natural pairing).

notations

We can define some double coset operators on this module. Here for simplicity we illustrate *tame* case. Let $\mathcal{U}_0$ and $\mathcal{U}_1$ denote two level subgroups of $G(\mathbb{A}_f)$. For any $x \in G(\mathbb{A}_f)$ with $x_p = 1$, we can define an R-linear map

$$[\mathcal{U}_0 x \mathcal{U}_1] : \mathcal{F}(G, \mathcal{U}_0) \longrightarrow \mathcal{F}(G, \mathcal{U}_1)$$

as follow: first decompose $\mathcal{U}_0 x \mathcal{U}_1$ into a finite disjoint union $\coprod_i \mathcal{U}_0 g_i$ and define

$$f|[\mathcal{U}_0 x \mathcal{U}_1] = \sum_i f|g_i.$$

In particular, if $U_0 = U_1$, this module endows a right action by tame Hecke algebras.
It is more subtle to define Hecke actions at p. The double coset operator is defined in the same way but x_p can only take value in a certain monoid.

notations

To define p-adic automorphic forms, we need to construct suitable Iw_p-module first. It is usually constructed by certain induction methods. We will use the following notations for induction:

If $B_1 \subset H_1$ are groups, R is a commutative ring, and $\chi : B_1 \longrightarrow R^*$ is a character, let

$$Ind_{B_1}^{H_1}\chi = \{f : H \longrightarrow R | f(hb) = f(h)\chi(b) \ for\ all\ h \in H_1,\ b \in B_1\}.$$

What's more, if Pro is a property for some functions $f \in Ind_{B_1}^{H_1}\chi$ that is invariant under left translation by H_1, then let

$$Ind_{B_1}^{H_1,Pro}\chi = \{f \in Ind_{B_1}^{H_1}\chi | f\ has\ property\ Pro\}.$$

Then $Ind_{B_1}^{H_1,P}$ is an R-module with a right action of H_1 given by left translations, i.e. $(f.h)(x) = f(hx)$ for all $h, x \in H_1$.

For example, let R denote a p-adic field, $\chi = (t_1, ..., t_n) \in \mathbb{Z}^n$ and we write $diag(d_1, .., d_n)$ for the diagonal matrix, we can interpret χ as the character of the diagonal torus $T(R)$ mapping $diag(d_1, ..., d_n)$ to $\prod_i d_i^{t_i}$ and thus also view it as the character of the upper triangular Borel subgroup $B(R)$ by reducing to $T(R)$ and applying χ. Assume $t_1 \geq t_2 ... \geq t_n$, the R-vector space

$$Ind_{B(R)}^{GL_n(R),alg} \chi,$$

where *alg* means *algebraic*, is the irreducible algebraic representation of GL_n over R with highest weight χ. Let w_0 denote the longest element of the Weyl group, then

$$\mathcal{F}(\mathcal{U}, Ind_{B(R)}^{GL_n(R),alg} w_0(-\chi))$$

is the space of classical (algebraic) automorphic forms on G of weight χ and level U with coefficients in R.

notations

More generally, a *weight* χ is a p-adic continuous character of $T(\mathbb{Z}_p) \cong (\mathbb{Z}_p^*)^n$. Or Equivalently, we can write χ as $(\chi_1, ..., \chi_n)$, each χ_i is a p-adic continuous character of $\mathbb{Z}_p^*$, and χ is defined by sending $diag(a_1, ..., a_n)$ to $\prod_i \chi_i(a_i)$. The *weight space* $\mathcal{W}$ is the rigid analytic space over $\mathbb{Q}_p$ such that for any $\mathbb{Q}_p$-affinoid algebra R, $\mathcal{W}(R)$ is the set of continuous characters $(\mathbb{Z}_p^*)^n \longrightarrow R^*$. Let $\triangle = ((\mathbb{Z}/p)^*)^n$ and we have

$$(\mathbb{Z}_p^*)^n \cong \triangle \times (1 + pZ_p)^n.$$

Thus any R-point of $\mathcal{W}$ is determined by a character of $\triangle$ and a character of $(1 + pZ_p)^n$. In geometry, the weight space is a finite disjoint union of n-dimensional open unit polydiscs (or called balls).

notations

For any $\mathbb{Q}_p$-affinoid algebra R, and a continuous character $\chi_1 : \mathbb{Z}_p^* \longrightarrow R^*$, it is called locally *r-analytic* (here $r = p^{-m}$ with m being a positive integer), if its restriction to $1 + p^m\mathbb{Z}_p$ can be given by a convergent power series with coefficient in R. For simplicity in this talk we will always work with radius r in the form of p^{-m}. Such convergent radius r for χ_1 always exists. For a weight $\chi = (\chi_1, ..., \chi_n)$, we call it locally r-analytic if each χ_i is locally r-analytic. Or equivalent, its restriction on $(1 + p^m\mathbb{Z}_p)^n$ is given by a convergent power series.

For any reduced $\mathbb{Q}_p$-affinoid X with a morphism $X \longrightarrow \mathcal{W}$, we use $[\cdot]_X$ to denote the resulting weight

$$[\cdot]_X : (\mathbb{Z}_p^*)^n \longrightarrow O(X)^*.$$

notations

Recall that we can also view the weight $[\cdot]_X$ as a character of $B(\mathbb{Z}_p)$, so we can consider the following induction $Ind_{B(\mathbb{Z}_p)}^{Iw_p}[\cdot]_X$. Notice that we have the following isomorphism (via natural inclusions)

$$\overline{N}(\mathbb{Z}_p) \cong Iw_p/B(\mathbb{Z}_p) \cong Iw_p B(\mathbb{Q}_p)/B(\mathbb{Q}_p),$$

so this $O(X)$-module has extra $B(\mathbb{Q}_p)$ right action and we can identify it (through restriction on $\overline{N}(\mathbb{Z}_p)$) with the space of $O(X)$-valued functions on $\overline{N}(\mathbb{Z}_p)$.
Moreover, we have the following identification

$$\mathbb{Z}_p^{\frac{n(n-1)}{2}} \cong \overline{N}(\mathbb{Z}_p),$$

$$\underline{z} = (z_{i,j}) \mapsto \overline{N}(\underline{z}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ pz_{2,1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ pz_{n,1} & \cdots & pz_{n,n-1} & 1 \end{pmatrix}.$$

notations

We put the dictionary order on the index set $Inx = \{(i,j)|1 \leq j < i \leq n\}$, so we can also present $\underline{z}$ by a tuple $(z_{2,1}, ..., z_{n,n-1})$. For later application in Ihara lemma, we further introduce the root subgroup $R_{2,1}$:

$$\mathbb{Z}_p \hookrightarrow \overline{N}(\mathbb{Z}_p),$$

$$a \mapsto \overline{N}((a, 0, \cdots, 0)).$$

It is easy to see that this is a group map, $R_{2,1}(a+b) = R_{2,1}(a)R_{2,1}(b)$. What's more, for any $\underline{z}$, set $\widetilde{z} = \underline{z} - (z_{2,1}, 0, \cdots, 0)$, then $\overline{N}(\underline{z}) = R_{2,1}(z_{2,1})N(\widetilde{z})$.

notations

For a function $f : \overline{N}(\mathbb{Z}_p) \longrightarrow O(X)$, we call it locally r-analytic (recall $r = p^{-m}$), if for any $\underline{a} = (a_{i,j}) \in \mathbb{Z}_p^{\frac{n(n-1)}{2}}$, the restriction of f on

$$Ball(\underline{a}, r) = \{\underline{z} = (z_{i,j}) \in \mathbb{Z}_p^{\frac{n(n-1)}{2}} | z_{i,j} \in a_{i,j} + p^m\mathbb{Z}_p\}$$

is given by a convergent power series with coefficients in $O(X)$.
Let $r - an$ denote the property being locally r-analytic, we introduce the $O(X)$ module

$$A_{X,r} = Ind_{B(\mathbb{Z}_p)}^{Iw_p, r-an}[\cdot]_X.$$

It still has the right Iw_p action. But be careful, the previous $B(\mathbb{Q}_p)$ action may not keep the property $r - an$. Instead there exists a monoid $\mathcal{M}$ of $B(\mathbb{Q}_p)$ that keeps the convergent radius r. Therefor $A_{X,r}$ has extra $\mathcal{M}$ action. Through this monoid action, we can define suitable Hecke operators at p and slope etc, which is very important for p-adic forms (e.g. see chapter 7 of [**?**]).

notations

To prove the abelian Ihara lemma, we explore more about $A_{X,r}$. From the definition, we have the following isomorphism

$$A_{X,r} \cong \oplus_k O(X)\langle Z_{i,j}\rangle,$$

this is a finite sum with $p^{m\frac{n(n-1)}{2}}$ components. This is done by pick up a set of representatives for $\mathbb{Z}_p^{\frac{n(n-1)}{2}}/(p^m\mathbb{Z}_p)^{\frac{n(n-1)}{2}}$, and identify the closed ball of radius r with the unit closed ball through rescaling. Thus as an $O(X)$-module, $A_{X,r}$ is just some copies of standard Tate algebras. For each summand, we write an element $Z \in O(X)\langle Z_{i,j}\rangle$ as

$$Z = \sum_{\alpha} x_\alpha Z^\alpha,$$

here $\alpha = (\alpha_{i,j})$ runs over $\mathbb{N}^{Inx}$, Z^α is short for $\prod_{i,j} Z_{i,j}^{\alpha_{i,j}}$, $x_\alpha \in O(X)$ and tends to 0. We define the space of *locally r-analytic* (or *r-overconvergent*) p-adic automorphic forms on G with level U and weight X as the following $O(X)$-module:

$$\mathcal{S}_X(\mathcal{U}, r) := \mathcal{F}(\mathcal{U}, A_{X,r}).$$

Now we turn to **dual** side.

For any Banach algebra R and Banach R-module A, we define the *dual module* A^* as the Banach R-module of continuous R-linear maps from A to R, with the usual operator norm. For example, let R be $\mathbb{Q}_p$ and A be $R\langle T\rangle$, the standard Tate algebra. Then its dual A^* can be identified with the module of power series with bounded coefficient $R[\langle T\rangle] = \mathbb{Z}_p[[T]] \otimes \mathbb{Q}_p$. In particular, A is an ONable module (has an orthonormal basis) while A^* is not.

For simplicity, we denote the $O(X)$-module $A^*_{X,r}$ as $D_{X,r}$. Notice that $A_{X,r}$ has an extra right actions by certain monoid $\mathcal{M}$ inside $B(\mathbb{Q}_p)$, we endow $D_{X,r}$ with right actions by $\mathcal{M}^{-1}$ through the dual action. In other words, for any $g \in \mathcal{M}$, $x \in A_{X,r}$ and $\lambda \in D_{X,r}$, we have

$$(x.g, \lambda) = (x, \lambda.g^{-1}),$$

here $(-,-)$ is the natural pairing.

What's more, in the previous section, we have explicit $A_{X,r}$ as a finite sum of standard Tate algebras, then correspondingly we have the following isomorphism:

$$D_{X,r} \cong \oplus_k O\langle X\rangle[\langle C_{i,j}\rangle].$$

Similarly, for each summand and an element C of it, we write

$$C = \sum_{\alpha} y_\alpha C^\alpha,$$

and the under the natural pairing we have

$$(Z, C) = \sum_{\alpha} x_\alpha y_\alpha.$$

We define the *dual space* of locally r-analytic p-adic forms on G with level $\mathcal{U}$ and weight X as the following $O(X)$-module

$$\mathcal{D}_X(\mathcal{U}, r) := \mathcal{F}(\mathcal{U}, D_{X,r}).$$

Later through a natural pairing, we can identify $\mathcal{D}_X(\mathcal{U}, r)$ with the $O(X)$-dual of $\mathcal{S}_X(\mathcal{U}, r)$.
Let S_0 denote a finite set of primes including p such that for any prime $q \notin S_0$, q is unramified in E and $G(\mathbb{Q}_q)$ is unramified over $\mathbb{Q}_q$ and U_q is a maximal open compact subgroup of $G(\mathbb{Q}_q)$. What's more, we always assume $\mathcal{U} = \mathcal{U}_{S_0} \times \prod_{q \notin S_0} \mathcal{U}_q$.
For any prime $q \notin S_0$, take any $x \in G(\mathbb{Q}_q)$ and let $T_{q,x}$ denote the Hecke operator corresponding to $\mathcal{U}_q x \mathcal{U}_q$ inside the $\mathbb{Z}$-valued local Hecke algebra $\mathcal{H}_q$. Let $T_{q,x}$ acts on $\mathcal{S}_X(\mathcal{U}, r)$ via the double coset operator $[\mathcal{U} x \mathcal{U}]$, and acts on $\mathcal{D}_X(\mathcal{U}, r)$ via $[\mathcal{U} x^{-1} \mathcal{U}]$.

notations

Pick up a suitable Hecke operator at p and let Fm denote its Fredholm series on $\mathcal{S}_X(\mathcal{U}, r)$. Suppose there is a decomposition $Fm = QR$ with Q and R are relative prime and Q is a polynomial with $Q(0) = 1$ and leading term is a unit. There is a canonical *slope decomposition*

$$\mathcal{S}_X(\mathcal{U}, r) = \mathcal{S}_X(\mathcal{U}, r)^Q \oplus Oth,$$

and $\mathcal{S}_X(\mathcal{U}, r)^Q$ is a finite projective $O(X)$-module and indeed independent of the (small enough) radius r. Moreover, this decomposition commutes with base change (fix $Fm = QR$), for any reduced affinoid $Y \longrightarrow X$, we have

$$\mathcal{S}_Y(\mathcal{U}, r)^Q \cong \mathcal{S}_X(\mathcal{U}, r)^Q \widehat{\otimes}_{O(X)} O(Y).$$

The Hecke algebra acts on $\mathcal{S}_X(\mathcal{U}, r)^Q$. Let X, r and Q vary, and glue the spectrum of image of Hecke algebras. Finally we get a rigid space (*eigenvariety*) $\mathcal{E}(\mathcal{U})$ with a weight map $\mathcal{E}(\mathcal{U}) \longrightarrow \mathcal{W}$.

Let det denote the natural map to abelian quotient $G \longrightarrow G^{ab}$ and we will also use it to denote the map $G(\mathbb{A}_f) \longrightarrow G^{ab}(\mathbb{A}_f)$ etc. Let Y be an irreducible reduced affinoid with a map $Y \longrightarrow \mathcal{W}$. In this section, we will prove the abelian Ihara lemma. It concerns about abelian forms, i.e. elements f of $\mathcal{S}_Y(\mathcal{U}, r)$ or $\mathcal{D}_Y(\mathcal{U}, r)$ that factors through (think it as a function on $G(\mathbb{A}_f)$) the abelian quotient $G^{ab}(\mathbb{A}_f)$ through the map det.
For any prime $q \notin S_0$, and an element $x_q \in G(\mathbb{Q}_q)$, consider the local Hecke operator T_{q,x_q} for the double coset $\mathcal{U}_q x_q \mathcal{U}_q$, we denote the number $deg(T_{q,x_q})$ for the cardinality of $(x_q^{-1}\mathcal{U}_q x_q \cap \mathcal{U}_q)\backslash\mathcal{U}_q$.
Now we can state the **abelian Ihara lemma**.

Abelian Ihara lemma

Proposition

Let Y be an irreducible reduced affinoid with a map $Y \longrightarrow \mathcal{W}$.
(1) If $\lambda \in \mathcal{D}_Y(\mathcal{U}, r)$ factors through the map det*, then $\lambda = 0$.*
(2) If $f \in \mathcal{S}_Y(\mathcal{U}, r)$ factors through the map det *and f is nonzero, write weight $[\cdot]_Y$ as $\chi = (\chi_1, ..., \chi_n)$, then χ is* **central***: $\chi_1 = \chi_2 = ... = \chi_n$, and there exists a finite etale cover $\widetilde{Y} \longrightarrow Y$, a finite abelian extension $E \longrightarrow \widetilde{E}$ and a p-adic continuous character*

$$\psi : G^{ab}(\mathbb{Q})\backslash G^{ab}(\mathbb{A}_f)/\det(\mathcal{U}^p) \longrightarrow O(\widetilde{Y})^*,$$

such that for almost all prime q of $\mathbb{Q}$ that splits in the field $\widetilde{E}$, the element $\psi(\pi_q)$ lies in $O(Y)^*$, and $T_{q,x_q}(f) = deg(T_{q,x_q})\psi(\pi_q)^{-v_q(\det(x_q))}f$. Moreover, the cover $\widetilde{Y}$, the field $\widetilde{E}$ and the map ψ only depends on Y.

Abelian Ihara lemma

These two statements are parallel and can be proved by similar ideas.
For the first statement, by definition, for any element $a \in G(\mathbb{A}_f)$ and $g \in \mathcal{U}$, we have $\lambda(a.g) = \lambda(a)g_p$. Pick up an element $g \in G^{der}(A_f) \cap \mathcal{U}$, we know

$$\lambda(a) = \lambda(a.g) = \lambda(a).g_p.$$

Notice that g_p can be any element in $SL(n, \mathbb{Q}_p) \cap Iw_p$ (just construct g with other components being trivial). Therefore for any $\widetilde{f} \in A_{X,r}$, we have

$$(\widetilde{f}.g_p^{-1} - \widetilde{f}, \lambda(a)) = 0.$$

This property will force λ to be zero,
For simplicity, by abuse of notations, still let λ denote $\lambda(a)$. Then it is an element in $D_{X,r}$. Recall that we have explicit this module as $\oplus_k O(Y)[\langle C_{i,j}\rangle]$, it is enough to show that each component of λ is zero, Take a summand $O(Y)[\langle C_{i,j}\rangle]$ and suppose the corresponding part of λ is C. Through **induction**, we will show that each C_α is zero.

Abelian Ihara lemma

We will do induction for the number $\alpha_{2,1}$. If it is 0, then consider $\widetilde{\alpha} = \alpha + (1, 0, \cdots, 0)$. The polynomial $Z^{\widetilde{\alpha}}$ certainly lies in the module $O(Y)\langle Z_{i,j} \rangle$ (view this module as the corresponding summand of $A_{X,r}$). Take an element $g_p = R_{2,1}(a_{2,1})$ with $a_{2,1} \in \mathbb{Z}_p$ with large enough valuation such that $Z^{\widetilde{\alpha}}.g_p$ still lies in this summand and suppose

$$Z^{\widetilde{\alpha}}.g_p = (Z_{2,1} + \delta) \prod_{(i,j) \neq (2,1)} Z^{\alpha_{i,j}}.$$

Then the difference $Z^{\widetilde{\alpha}}.g_p - Z^{\widetilde{\alpha}}$ is exactly δZ^{α}. By the above property, $(\delta Z^{\alpha}, C) = 0$. Then $C_{\alpha} = 0$.

Suppose for any α with $\alpha_{2,1} < N_0$ (N_0 is a positive integer), we have $C_{\alpha} = 0$. Now for an element α with $\alpha_{2,1} = N_0$, we do the above process again to get $\widetilde{\alpha}$, g_p and

$$Z^{\widetilde{\alpha}}.g_p = (Z_{2,1} + \delta)^{N_0+1} \prod_{(i,j) \neq (2,1)} Z^{\alpha_{i,j}}.$$

Notice that $(Z_{2,1} + \delta)^{N_0+1} - Z_{2,1}^{N_0+1} = \delta Z_{2,1}^{N_0} + \sum_{e<N_0} \delta_e Z_{2,1}^{e}$, then $C_{\alpha} = 0$.

Abelian Ihara lemma

Now let's turn to the second statement.
Still pick up any $g \in G^{der}(\mathbb{A}_f) \cap \mathcal{U}$ and $x \in G(\mathbb{A}_f)$, we get

$$f(x) = f(x.g) = f(x).g_p.$$

For any $x_0 \in G(\mathbb{A}_f)$ with $f(x_0) \neq 0$ (such x_0 exists because f is nonzero), set $\widetilde{f} = f(x_0)$, an element in $A_{X,r}$. Then $\widetilde{f}$ is invariant under the group $SL(n, \mathbb{Q}_p) \cap Iw_p$. Then the Iw_p-action factors through the abelian quotient $\det : Iw_p \twoheadrightarrow \mathbb{Z}_p^*$. Denote the kernel of det as Iw_p^{der}. Notice that $Iw_p = Iw_p^{der} T(\mathbb{Z}_p)$, for any $n_0 \in \overline{N}(\mathbb{Z}_p)$ and $b_0 \in T(\mathbb{Z}_p)$, we have

$$\widetilde{f}(n_0.b_0) = (\widetilde{f}.n_0)(b_0) = \widetilde{f}(b_0) = \chi(b_0)\widetilde{f}(1).$$

In particular, $\widetilde{f}(1) \neq 0$. For any $t \in \mathbb{Z}_p^*$, set $b = diag(t, t^{-1}, 1, ..., 1)$, then

$$\widetilde{f}(1) = \widetilde{f}(b) = \chi_1(t)\chi_2(t)^{-1}\widetilde{f}(1).$$

Because $O(Y)$ is a domain, we get $\chi_1 = \chi_2$. Similarly, we find

$$\chi_1 = \chi_2 = ... = \chi_n.$$

Abelian Ihara lemma

We have showed that the weight χ is central.
What's more, the det map has a (non-canonical) section $s : \mathbb{Z}_p^* \longrightarrow Iw_p$, sending $a \in \mathbb{Z}_p^*$ to $diag(a, 1, ..., 1)$. Then we can write down the Iw_p action more clearly, for any $g \in Iw_p$, we have

$$\widetilde{f}.g = \chi_1(\det(g))\widetilde{f}.$$

Abelian Ihara lemma

Notice that the map $\det : G \longrightarrow G^{ab}$ between two group schemes over $\mathbb{Q}$ also has a non-canonical section via picking up an anisotropic vector inside the Hermitian space over E, then the map $G(\mathbb{Q}) \longrightarrow G^{ab}(\mathbb{Q})$ is surjective, and we have $G^{ab}(\mathbb{Q}) \cap \det(\mathcal{U}) = 1$ (we can also shrink the level $\mathcal{U}$ in the beginning to get this property).
Consider the p-adic continuous character $\psi_0 : \det(\mathcal{U}) \longrightarrow \det(\mathcal{U}_p) = Z_p^* \longrightarrow O(Y)^*$, since $\det(\mathcal{U}) \cap G^{ab}(\mathbb{Q}) = 1$, we get a p-adic character (again denote it by ψ_0):

$$G^{ab}(\mathbb{Q}) \backslash \det(\mathcal{U}) / \det(\mathcal{U}^p) \longrightarrow O(Y)^*.$$

Notice the source is a subgroup of $G^{ab}(\mathbb{Q}) \backslash G^{ab}(\mathbb{A}_f) / \det(\mathcal{U}^p)$ with finite index (again by generalized finiteness of class group), therefore there exists a finite etale cover $\widetilde{Y} \longrightarrow Y$ and a p-adic continuous character

$$\psi : G^{ab}(\mathbb{Q}) \backslash G^{ab}(\mathbb{A}_f) / \det(\mathcal{U}^p) \longrightarrow O(\widetilde{Y})^*$$

extending the original character ψ_0.

Abelian Ihara lemma

Recall $G^{ab} = U(1)$, now apply class field theory to the field E, we can find a finite abelian extension $E \longrightarrow \widetilde{E}$ with a canonical isomorphism

$$Gal(\widetilde{E}/E) \cong G^{ab}(\mathbb{Q})\backslash G^{ab}(\mathbb{A}_f)/\det(\mathcal{U}).$$

In particular, for any prime q of $\mathbb{Q}$ that splits in $\widetilde{E}$, we know that π_q lies in $G^{ab}(\mathbb{Q})\det(\mathcal{U})$.

Further assume that $q \notin S_0$, let's compute the Hecke action T_{q,x_q} $(x_q \in G(\mathbb{Q}_q))$. First observe that

$$\mathcal{U}x_q\mathcal{U} = \coprod_i \mathcal{U}x_q y_i, \;\; y_i \in (x_q^{-1}\mathcal{U}_q x_q \cap \mathcal{U}_q)\backslash\mathcal{U}_q,$$

therefore for any $g \in G(\mathbb{A}_f)$, we have

$$\begin{aligned} T_{q,x_q}(f)(g) &= \sum_i (f|_{x_q y_i})(g) \\ &= \sum_i f(g y_i^{-1} x_q^{-1}) = \sum_i f(g x_q^{-1} y_i^{-1}) \\ &= \sum_i ((f|y_i)|x_q)(g) = deg(T_{q,x_q})(f|x_q)(g). \end{aligned}$$

Here the last equality is because f is invariant under $\mathcal{U}$-action.
Therefore it remains to compute more explicitly the element

$$(f|x_q)(g) = f(g x_q^{-1}).$$

Abelian Ihara lemma

By our assumption on q, we know that $\det(\pi_q) \in G^{ab}(\mathbb{Q})\det(\mathcal{U})$. On the other hand, $\det(x_q) = \pi_q^{v_q(\det(x_q))}\varepsilon$ with $\varepsilon \in Z_q^*$, while $\det(\mathcal{U}_q) = \mathbb{Z}_q^*$, we find that

$$\det(x_q) \in G^{ab}(\mathbb{Q})\det(\mathcal{U}).$$

Then $\psi(\det(x_q)) = \psi_0(\det(x_q)) \in O(Y)^*$.
Notice that $\det : G(\mathbb{Q}) \longrightarrow G^{ab}(\mathbb{Q})$ is surjective, therefore we can find $x_0 \in G(\mathbb{Q})$ and $x_1 \in \mathcal{U}$ such that $\det(x_q)^{-1} = \det(x_0)\det(x_1)$. Now we have

$$f(gx_q^{-1}) = f(gx_0x_1) = f(x_0gx_1)$$

$$= f(gx_1) = f(g).x_{1,p} = \chi_1(\det(x_{1,p}))f(g),$$

here we use the property that f is abelian and f factors through $G(\mathbb{Q})\backslash G(\mathbb{A}_f)$.
Finally from the construction of ψ, we get

$$\chi_1(\det(x_{1,p})) = \psi(\det(x_q))^{-1} = \psi(\pi_q)^{-v_q(\det(x_q))},$$

Abelian Ihara lemma

in summary,

$$T_{q,x_q}(f) = deg(T_{q,x_q})\psi(\pi_q)^{-v_q(\det(x_q))}f.$$

Moreover, the construction of $\widetilde{E}$, $\widetilde{Y}$ and ψ only depends on Y (independent of f). We're done.
Indeed this theorem holds more generally, at least for those reductive group G over $\mathbb{Q}$ such that $G(\mathbb{R})$ is compact and G^{der} is simply connected. In the proof of this theorem, we haven't used too much special properties of unitary groups and many statements holds in the later general setting. For example, we use a special root subgroup $R_{2,1}$, but indeed the only property we need is that this root subgroup **commutes** with other root subgroups. Through the commutator relations between root subgroups, we can always find such a root (e.g. highest root is suitable for us).

Abelian Ihara lemma

Another special fact during the proof is that we use certain non-canonical section to show $G(\mathbb{Q}) \twoheadrightarrow G^{ab}(\mathbb{Q})$. Now we illustrate a general argument through **Galois cohomology.** Consider the exact sequence

$$1 \longrightarrow G^{der} \longrightarrow G \longrightarrow G^{ab} \longrightarrow 1.$$

Because G^{der} is simply connected, for any non-Archimedean field k, the Galois cohomology $H^1(k, G^{der})$ vanish and the Hasse principle holds, the map $H^1(\mathbb{Q}, G^{der}) \longrightarrow \prod_v H^1(\mathbb{Q}_v, G^{der})$ (v runs over all place) is injective (indeed, it is bijective). We only need to care about $H^1(\mathbb{R}, G^{der})$ now. Because $G^{ab}(\mathbb{R})$ is a connected compact lie group (indeed isomorphic to products of $U(1)(\mathbb{R})$), the map $G(\mathbb{R}) \longrightarrow G^{ab}(\mathbb{R})$ is surjective. Then use the above exact sequence, we get an injection $H^1(\mathbb{R}, G^{der}) \hookrightarrow H^1(\mathbb{R}, G)$. Combine these results together, we get $H^1(\mathbb{Q}, G^{der}) \hookrightarrow H^1(\mathbb{Q}, G)$. Therefore the map $G(\mathbb{Q}) \longrightarrow G^{ab}(\mathbb{Q})$ is surjective in general. And we can similarly construct ψ, $\widetilde{Y}$ and $\widetilde{E}$ etc in the later general setting.

pairing

Still Let $X \longrightarrow \mathcal{W}$ denote a reduced affinoid. Now We introduce the natural pairing between $\mathcal{S}_X(\mathcal{U}, r)$ and $\mathcal{D}_X(\mathcal{U}, r)$.
Take a $\mathbb{Q}$-valued Haar measure $\mu_{\mathcal{U}}$ on $G(\mathbb{Q})\backslash G(\mathbb{A}_f)$ with $\mu_{\mathcal{U}}(\mathcal{U}) = 1$. Then we define the pairing as follow:

$$(f, \lambda) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A}_f)} (f(g), \lambda(g)) d\mu_{\mathcal{U}},$$

where $f \in \mathcal{S}_X(\mathcal{U}, r)$ and $\lambda \in \mathcal{D}_X(\mathcal{U}, r)$. More explicitly, we can write this integration as a finite sum. Fix a set of double coset representatives $\{x_i | 1 \leq i \leq h\}$ for $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/\mathcal{U}$, then we can rewrite the pairing as (use neatness of $\mathcal{U}$)

$$(f, \lambda) = \sum_{i=1}^{h} (f(x_i), \lambda(x_i)).$$

pairing

Recall that through this set of representatives, we have isomorphisms $\mathcal{S}_X(\mathcal{U}, r) \cong A^h_{X,r}$ and $\mathcal{D}_X(\mathcal{U}, r) \cong D^h_{X,r}$, then through this natural pairing, we can identify $\mathcal{D}_X(\mathcal{U}, r)$ with the $O(X)$-dual of $\mathcal{S}_X(\mathcal{U}, r)$ (thus justify the name *dual*). Moreover, regarding the slope decomposition

$$\mathcal{S}_X(\mathcal{U}, r) = \mathcal{S}_X(\mathcal{U}, r)^Q \oplus Oth,$$

we define $\mathcal{D}_X(\mathcal{U}, r)^Q$ to be the submodule of $\mathcal{D}_X(\mathcal{U}, r)$ consists of maps that sends Oth to 0. Then it is naturally the dual module of $\mathcal{S}_X(\mathcal{U}, r)^Q$.

pairing

Through the identification of $\mathcal{D}_X(\mathcal{U}, r)$ with the $O(X)$-dual of $\mathcal{S}_X(\mathcal{U}, r)$, its Hecke action is also the dual action:

Proposition

For any $f \in \mathcal{S}_X(\mathcal{U}, r)$, $\lambda \in \mathcal{D}_X(\mathcal{V}, r)$ and $g \in G(\mathbb{A}_f)$ with $g_p \in Iw_p$, we have $(f|[\mathcal{U}g\mathcal{V}], \lambda) = (f, \lambda|[\mathcal{V}g^{-1}\mathcal{U}])$.

We can prove it via explicit computations. And it still holds if the level subgroup is not neat.

From now on, we assume that $n = 3$. And we specify three level groups to study level raising questions.
Pick up a level subgroup $\mathcal{U}_0$, recall that we have defined a finite set S_0 of ('bad') primes. Take an odd prime l that is inert in E and $l \notin S_0$. Then $G(\mathbb{Q}_l)$ is an unramified unitary group $U(3)(\mathbb{Q}_l)$ and it is rank one. We further assume that the l-part $\mathcal{U}_{0,l}$ is a hyperspecial subgroup of $G(\mathbb{Q}_l)$, thus corresponding to a hyperspecial vertex in the Bruhat-Tits tree for $G(\mathbb{Q}_l)$. Let T_l denote the standard local Hecke operator at l (generator for this spherical Hecke algebra). Let $\widetilde{\mathcal{U}}_l$ denote the maximal compact open subgroup of $G(\mathbb{Q}_l)$ corresponding to an adjacent vertex. Then $\widetilde{\mathcal{U}}_l \cap \mathcal{U}_{0,l}$ is an Iwahori subgroup. Let $S_1 = S_0 \cup \{l\}$, and we get two other level subgroups $U_1 = U_{S_0} \times \widetilde{\mathcal{U}}_l \times \mathcal{U}_0^{S_1}$ and $V = U_0 \cap U_1$. These three level subgroups only differ at the l-part. And we let $\mathbb{T}^{S_0}$ denote the integral abstract tame Hecke algebras away from S_0. And $\mathbb{T}^{S_1}$ denote similarly the abstract tame Hecke algebras away from S_1.

old and new forms

To simplify notations, we introduce

$$L_0 = \mathcal{S}_X(\mathcal{U}, r)^Q,\ \ L_0^* = \mathcal{D}_X(\mathcal{U}, r)^Q,$$

$$L_1 = \mathcal{S}_X(\mathcal{U}_1, r)^Q,\ \ L_1^* = \mathcal{D}_X(\mathcal{U}, r)^Q,$$

$$M = \mathcal{S}_X(\mathcal{V}, r)^Q,\ \ M^* = \mathcal{D}_X(\mathcal{V}, r)^Q.$$

Now we introduce the '*level raising*' map $i : L_0 \oplus L_1 \longrightarrow M$ by

$$i(f_0, f_1) = f_0|[\mathcal{U}_0 1 \mathcal{V}] + f_1|[\mathcal{U}_1 1 \mathcal{V}].$$

We define the image $im(i)$ inside M as the space of *old (at l) forms*.
We also have a '*level lowering*' map $i^+ : M \longrightarrow L_0 \oplus L_1$ by

$$i^+(f) = (f|[\mathcal{V} 1 \mathcal{U}_0], f|[\mathcal{V} 1 \mathcal{U}_1]).$$

We define its kernel $ker(i^+)$ as the space of *new (at l) forms*.

Use the same double coset operators, we get maps in the dual side:

$$j : L_0^* \oplus L_1^* \longrightarrow M^*,$$

$$j^+ : M^* \longrightarrow L_0^* \oplus L_1^*.$$

In the classical setting, the space of automorphic forms with fixed level and weight is finite dimensional and self dual under the previous pairing, the map i^+ is the dual map for i. But here the map j^+ is the dual map for i, and the map j is the dual map for i^+:

$$(i(f_0, f_1), \lambda) = ((f_0, f_1), j^+(\lambda)),$$

$$(i^+(f), (\lambda_0, \lambda_1)) = (f, j(\lambda_0, \lambda_1)).$$

We first compute the composition $i^+ \circ i$. It is an endomorphism of $L_0 \oplus L_1$ by a ('level changing') matrix (acts from the right):

$$\begin{pmatrix} l^3+1 & [\mathcal{U}_0 1 \mathcal{U}_1] \\ [\mathcal{U}_1 1 U_0] & l+1 \end{pmatrix}.$$

The computation of these double coset operators is purely local about the place l. And each operator can be interpreted by some combinatorial operators about the Bruhat-Tits tree.

In addition, the composition of the following map

$$L_0 \xrightarrow{[\mathcal{U}_0 1 \mathcal{U}_1]} L_1 \xrightarrow{[\mathcal{U}_1 1 \mathcal{U}_0]} L_0$$

is exactly

$$T_l + (l^3+1).$$

The results in the dual side are the same.

The first step is to establish the following result:

Proposition

If $X \hookrightarrow \mathcal{W}$ is an irreducible reduced affinoid which is admissible open in $\mathcal{W}$, then the map $i^+ \circ i$ is injective.

injection

Let $\widetilde{L}$ denote the $O(X)$-module $ker(i^+ \circ i)$ and L denote its image inside L_0 under the first projection.
If $\widetilde{L} \neq 0$, for any $(f_0, f_1) \in \widetilde{L}$, we have

$$(l^3 + 1)f_0 + f_1|[\mathcal{U}_1 1 \mathcal{U}_0] = 0,$$

$$f_0|[\mathcal{U}_0 1 \mathcal{U}_1] + (l + 1)f_1 = 0.$$

Combine them together, we get

$$(T_l - l(l^3 + 1))f_0 = 0.$$

In particular, the Hecke operator T_l acts on L through the scalar $l(l^3 + 1)$.
Apply the general machine of (local) eigenvariety and some dimension argument, the resulting closed space (via L) is a union of irreducible components of $\mathcal{E}(\mathcal{U}_0)$. In particular, the Hecke operator $T_l - l(l^3 + 1)$ vanishes on these irreducible components. Next we will show this is impossible.

injection

For any classical point z_0 at such irreducible components, let f denote a classical form corresponding to this point. Then we can canonically realize f inside $\oplus_k \Pi_k$, where this is a finite sum and each Π_k is an automorphic representation of $G(\mathbb{A}_f)$. Assume that $\Pi_k = \Pi_{k,l} \otimes \widehat{\Pi_{k,l}}$, where $\Pi_{k,l}$ is the corresponding l-part, an irreducible representation of $G(\mathbb{Q}_l)$. As the eigenvalue of T_l is $l(l^3+1)$, this unramified representation is a character. Then f is invariant under right multiplication by $G^{der}(\mathbb{Q}_l)$. Because f is a continuous function on $G(\mathbb{A}_f)$ and it is left invariant under multiplication by $G(\mathbb{Q})$, apply the strong approximation theorem to G^{der}, this function f is invariant under $G^{der}(\mathbb{A}_f)$ action. Then we can apply the abelian Ihara lemma, the abelian form f has central weight. But such locus is only a one dimensional Zaraski closed subspace inside the whole three dimensional weight space. It contradicts to Zaraski density of classical points on the eigenvariety. Therefore i^+i is injective.

Let $F(X)$ denote the fraction field of $O(X)$. For the $O(X)$-module L_0, for simplicity we will use $L_{0,F(X)}$ to denote $L_0 \otimes_{O(X)} F(X)$ and similarly for other $O(X)$-modules.
The injectivity of $i^+ \circ i$ implies the injectivity of $j^+ \circ j$. And we further introduce some auxiliary modules and construct other pairings:

$$\gamma_0 = L_0 \oplus L_1, \ \widetilde{\gamma_0} = L_0^* \oplus L_1^*;$$

$$\gamma_1 = i^+(M), \ \widetilde{\gamma_1} = j^+(M^*);$$

$$\gamma_2 = i^+(M \cap i(\gamma_{0,F(X)})), \ \widetilde{\gamma_2} = j^+(M^* \cap j(\widetilde{\gamma_0}_{F(X)}));$$

$$\gamma_3 = i^+ i(\gamma_0), \ \widetilde{\gamma_3} = j^+ j(\widetilde{\gamma_0}).$$

duality

Combine with perfect pairing $\gamma_{0,F(X)} \times \widetilde{\gamma}_{0F(X)} \longrightarrow F(X)$ and the injection j we find

$$(\gamma_1)^* \cong M^* \cap j(\widetilde{\gamma}_{0F(X)})$$

and these modules are reflexive. Then we get the following perfect pairing

$$P_1 : \gamma_0/\gamma_1 \times \frac{M^* \cap j(\widetilde{\gamma}_{0F(X)})}{j(\widetilde{\gamma}_0)} \longrightarrow F(X)/O(X).$$

On the dual side, we get the second perfect pairing:

$$P_2 : \frac{M \cap i(\gamma_{0,F(X)})}{i(\gamma_0)} \times \widetilde{\gamma}_0/\widetilde{\gamma}_1 \longrightarrow F(X)/O(X).$$

Finally combine with the perfect pairing on $M \times M^*$, we get the third pairing

$$P_3 : ker(i^+) \times \frac{M^*}{M^* \cap j(\widetilde{\gamma}_{0F(X)})} \longrightarrow O(X),$$

which identify $ker(i^+)$ with the $O(X)$-dual module of $\frac{M^*}{M^* \cap j(\widetilde{\gamma}_{0F(X)})}$.

Let $\mathbf{H}$ denote the image of $O(X) \otimes \mathbb{T}^{S_1}$ in $End_{O(X)}(M)$. Let M_0 denote an $\mathbf{H}$-module which is finitely generated over $O(X)$. We call M_0 is *very Eisenstein* if for each prime ideal $\mathfrak{b}$ of $\mathbf{H}$ in the support of M_0 with $\mathfrak{p} = \mathfrak{b} \cap O(X)$, the resulting Hecke module satisfies the second statement in the abelian Ihara lemma.
The abelian Ihara lemma implies the following result:

Lemma

Let $Y \hookrightarrow X$ be a closed, reduced and irreducible sub-affinoid. Then the module $Tor_1^{O(X)}(M/i(\gamma_0), O(Y))$ is very Eisenstein and $Tor_1^{O(X)}(M^*/j(\widetilde{\gamma}_0), O(Y))$ is zero.

very Eisenstein

Through some commutative algebra arguments about support, we can upgrade it into the following form:

Lemma

(1) The module $(M/i(\gamma_0))^{tors}$ is very Eisenstein.
(2) The module $(M^*/j(\widetilde{\gamma}_0))^{tors}$ is 0.

Apply previous duality results, we get

Proposition

The modules γ_2/γ_3 and $\widetilde{\gamma}_0/\widetilde{\gamma}_1$ are very Eisenstein. The modules γ_0/γ_1 and $\widetilde{\gamma}_2/\widetilde{\gamma}_3$ are 0.

Let $\mathbf{H}_0$ denote the image of $\mathbb{T}^{S_0} \otimes O(X)$ inside $End_{O(X)}(L_0)$. Through the embedding $L_0 \hookrightarrow L_0 \oplus L_1 \hookrightarrow M$ (the first map is just the natural inclusion), we get a finite map (via restriction) $\mathbf{H} \longrightarrow \mathbf{H}_0$. For any ideal I of $\mathbf{H}_0$, let I_M denote the inverse image of I in $\mathbf{H}$. For any finite $\mathbf{H}_0$-module M_0, if a prime ideal $\mathbf{p}$ lies in $supp_{\mathbf{H}_0}(M_0)$, then $\mathbf{p}_M$ also lies in $supp_{\mathbf{H}}(M_0)$.
We have the following proposition concerning p-adic level raising:

Proposition

Suppose $\mathbf{p}$ *is a prime ideal of* $\mathbf{H}_0$ *such that* $\mathbf{p}_M$ *is not very Eisenstein. If we further assume that* $\mathbf{p}$ *contains* $T_l - l(l^3 + 1)$, *then* $\mathbf{p}_M$ *lies in the support of* $\mathbf{H}$*-module* $ker(i^+)$.

level raising

Consider the **H**-module $Q = \widetilde{\gamma}_0/\widetilde{\gamma}_3$. Recall that the composition j^+j can be represented by a matrix (acts from right)

$$\begin{pmatrix} l^3+1 & [\mathcal{U}_0 1 \mathcal{U}_1] \\ [\mathcal{U}_1 1 U_0] & l+1 \end{pmatrix}.$$

Consider the following **H**-equivariant surjection

$$\rho : \widetilde{\gamma}_0 \twoheadrightarrow L_0^*,$$

$$(f_0, f_1) \mapsto -(l+1)f_0 + f_1|[\mathcal{U}_1 1 \mathcal{U}_0].$$

After localization, the following composition is still surjective:

$$\widetilde{\gamma}_{0\mathbf{p}_M} \twoheadrightarrow \widetilde{\gamma}_{0\mathbf{p}_M} \twoheadrightarrow L^*_{0,\mathbf{p}_M}.$$

However, this composition is represented by the matrix $\begin{pmatrix} T_l - l(l^3+1) \\ 0 \end{pmatrix}$.

level raising

In particular, it will imply that $T_l - l(l^3+1)$ is an surjective endomorphism for $L^*_{0,\mathbf{p}_M}$. Now we consider the $\mathbf{H}_0$-action on L^*_0 and further localize to $\mathbf{p}$, we find that the map $T_l - l(l^3+1)$ is a surjective endomorphism for a finitely generated nonzero module $L^*_{0,\mathbf{p}}$. But $\mathbf{p}$ contains $T_l - l(l^3+1)$, such multiplication can't be surjective due to Nakayama's lemma. Therefore $\mathbf{p}_M \in supp_{\mathbf{H}}(Q)$.
any prime ideal lying in the support of $\widetilde{\gamma}_0/\widetilde{\gamma}_1$ or $\widetilde{\gamma}_2/\widetilde{\gamma}_3$ is very Einsenstein. Therefore $\mathbf{p}_M$ must lie in the support of $\widetilde{\gamma}_1/\widetilde{\gamma}_2$. Then $\mathbf{p}_M$ also lies in the support of $\frac{M^*}{M^*\cap j(\widetilde{\gamma}_{0F(X)})}$.
Through the pairing P_3, we conclude that $\mathbf{p}_M$ lies in the support of $ker(i^+)$.

Now we apply this result to study the geometry of eigenvarieties.
Let $\mathcal{E}(\mathcal{V})$ denote the (reduced) eigenvariety with level $\mathcal{V}$, constructed via the Hecke algebra $\mathbb{T}^{S_1} \otimes \mathcal{H}_p^-$. Similarly use Hecke algebra $T^{S_0} \otimes \mathcal{H}_p^-$ to construct the (reduced) eigenvariety $\mathcal{E}(\mathcal{U}_0)$ with level $\mathcal{U}_0$.
Apply the general machine of eigenvariety to $im(i)$ (old forms), we get a closed immersion

$$\mathcal{E}(\mathcal{V})^{old} \hookrightarrow \mathcal{E}(\mathcal{V}).$$

Apply some dimension arguments, it is a union of irreducible components. Roughly speaking, it is the Zaraski closure of classical points corresponding to old forms.
In a similarly, through $ker(i^+)$ (new forms), we get a closed immersion

$$\mathcal{E}(\mathcal{V})^{new} \hookrightarrow \mathcal{E}(\mathcal{V}),$$

which is a union of irreducible components and also the Zaraski closure of classical points corresponding to new forms.

Moreover, as a classical form with level $\mathcal{V}$ is either old at l or new at l, apply the Zaraski density of classical points, we divide irreducible components of $\mathcal{E}(\mathcal{V})$ into two types.

Through the inclusion $\mathcal{S}_X(\mathcal{U}_0, r) \xrightarrow{[\mathcal{U}_0 1 \mathcal{V}]} \mathcal{S}_X(\mathcal{V}, r)$ we have a natural finite map

$$\mathcal{E}(\mathcal{U}_0) \longrightarrow \mathcal{E}(\mathcal{V})^{old}.$$

And we can tranlate the previous results into the following theorem:

Theorem

Suppose we have a point ϕ on $\mathcal{E}(\mathcal{U}_0)$ which is not very Eisenstein and satisfies $T_l(\phi) = l(l^3 + 1)$. Then the corresponding point inside $\mathcal{E}(\mathcal{V})^{old}$ will also lie in $\mathcal{E}(\mathcal{V})^{new}$.

How to construct these points?
Suppose the vanishing locus of the Hecke operator $T_l - l(l^3+1)$ is non-empty, then it is a codimension one (thus dimension two) space inside $\mathcal{E}(\mathcal{U}_0)$. Because each classical point inside this vanishing locus is very Eisenstein (in particular has central weight). After cutting out the (at most one dimensional) closed subspace over central weights inside this locus, the resulting space is still non-empty and two dimensional. This space consists of non-classical points and lies in the intersection of old components and new components by this theorem.
Therefore the key point is to construct suitable eigenvariety such that the vanishing locus of the Hecke operator $T_l - l(l^3+1)$ is non-empty.

further development

For definite quaternion algebras over $\mathbb{Q}$, Newton constructed such intersection points by some explicit (delicate) computations about Hida families (via Sage). In $U(3)$ setting, such explicit computation about ordinary Hecke algebras is harder.

Instead I'm trying to apply the p-adic Langlands functoriality. Because $GL(2)$ is also closely related with $U(2)$, the first step is to transfer his result into definite $U(2)$ eigenvarieties. Then apply the p-adic symmetric square functoriality, we can get the desired exotic points on the definite $U(3)$ eigenvariety. Because all points are non-classical, we have to do p-adic interpolation of classical functoriality. For example, through the method in Hansen's paper [**?**]. I'm also thinking about p-adic Jacquet-Langlands between different unitary groups by this method. I hope to finish these details soon.

further development

On the other hand, as we mentioned before, the abelian Ihara lemma holds for reductive G over $\mathbb{Q}$ with $G(\mathbb{R})$ being compact and G^{der} being simply connected. If further there exists a prime l such that $G(\mathbb{Q}_l)$ has reduced rank one, then the main theorem also holds. But it is much more difficult to deal with higher rank cases. Indeed, even in the classical setting, the conjecture by Clozel, Harris and Taylor about Ihara lemma (of GL_n) is still open.

References

Thank you!