FROM GAUSS SUMS TO MODERN NUMBER THEORY

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The main goal of this seminar is to provide a classical introduction to modern number theory. More concretely, we will begin with classical objects such as Gauss sums and gradually approach more modern topics. For each speaker, the emphasis should be on key ideas, intuitive motivations, and illustrative toy examples; technical details may be omitted when appropriate.

The main references are

- Serre's ***A Course in Arithmetic***,
- Ireland–Rosen's ***A Classical Introduction to Modern Number Theory***,
- Neukirch's ***Algebraic Number Theory***.

> *Remark:* These books contain many general results; Instead speakers are encouraged to focus on toy examples to avoid unnecessary technicalities.

Gauss sums. We first recall Gauss sums. See Serre's book Chapter 1, for more details.

Let p be an odd prime. Let ψ denote the additive character of $\mathbb{F}_p$ given by $\psi(a) = \zeta_p^a$, where ζ_p is a primitive pth root of unity in $\mathbb{C}$. Let χ denote the quadratic character of $\mathbb{F}_p^\times$, i.e., $\chi(a) = \left(\frac{a}{p}\right)$, the nontrivial character of $\mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2$. Extend χ to a function on $\mathbb{F}_p$ by setting $\chi(0) = 0$. The Gauss sum is defined as

$$G(\chi, \psi) = \sum_{a \in \mathbb{F}_p} \chi(a)\psi(a).$$

A basic property is

$$G(\chi, \psi)^2 = \chi(-1)p.$$

Now for any odd prime $\ell \neq p$, using reduction modulo ℓ and the Frobenius automorphism in $\overline{\mathbb{F}}_\ell$, one obtains the quadratic reciprocity law:

$$\left(\frac{\ell}{p}\right)\left(\frac{p}{\ell}\right) = (-1)^{\frac{p-1}{2}\frac{\ell-1}{2}}.$$

This proof is ingenious yet not entirely intuitive. In fact, it represents the tip of the iceberg, offering numerous entry points into modern number theory. We will explore the following topics.

- **First topic:** an illustrative explanation of above proof of quadratic reciprocity via Gauss sums, through which we also gain an introduction to algebraic number theory. We will see that quadratic reciprocity is a shadow of class field theory, itself the simplest instance of the Langlands program.
- **Second topic:** using Gauss sums to prove the Weil conjectures, a starting point for modern algebraic and arithmetic geometry. We will take a glimpse at the seminal ideas that shaped Weil's insight.
- **Third topic:** Dirichlets theorem, which illustrates the power of L-functions and harmonic analysis.
- **Fourth topic:** theta functions and zeta functions. By employing the functional equation of theta series, we can determine the Gauss sum itself (beyond merely its

square). We will also discuss the theory of special values, which plays a significant role in modern number theory.
- **Fifth topic:** p-adic zeta functions, building on the fourth topic—specifically, on special values of zeta functions. The remarkable picture here is that we can do p-adic interpolation, which offers a glimpse into the fascinating world of number theory.

More detailed outlines for each topic are as follows.

Topic 1: Quadratic reciprocity. In this topic, we will introduce basic algebraic number theory, guided by the following picture:

$$\text{Number fields} \longleftrightarrow p\text{-adic fields} \longleftrightarrow \text{Finite fields.}$$

We will focus on quadratic extensions $\mathbb{Q}(\sqrt{d})$ and cyclotomic extensions $\mathbb{Q}(\zeta_p)$. The speaker may follow Neukirch, Chapter 1, Section 10 for number fields and Chapter 2 for p-adic fields.

The starting point of class field theory is to seek an "inner" description of the "outer" Galois group (e.g., describing prime decomposition). Cyclotomic extensions provide a nice example of reciprocity. Ignoring the prime 2, the extension $\mathbb{Q}(\zeta_p)$ ramifies only at p (indeed p is totally ramified); any other prime ℓ is unramified. Moreover, the corresponding Frobenius element Frob_ℓ sends ζ_p to ζ_p^ℓ.

Such reciprocity laws can be proved by studying p-adic fields. After introducing basic results on p-adic fields—Hensel's lemma, extension of valuations, and Newton polygons—it becomes easy to study local cyclotomic extensions and verify the above properties.

We then obtain a natural proof of the quadratic reciprocity law (Neukirch, Proposition 10.5). The Gauss sum embeds the quadratic field $\mathbb{Q}\big(\sqrt{(\frac{-1}{p})p}\big)$ into the cyclotomic field $\mathbb{Q}(\zeta_p)$; using the cyclotomic reciprocity described above (i.e., studying prime decomposition and Frobenius), the quadratic reciprocity law follows immediately.

If time permits, the speaker may also mention Exercise 1 in Section 10, Chapter 1 of Neukirch, which states that there are infinitely many primes ℓ congruent to 1 modulo p. This provides a special case of Dirichlet's theorem (Topic 3), although the general proof requires analytic methods.

> *Remark:* Many concepts cannot be covered in depth. Cyclotomic fields play a central role in number theory: class field theory shows that the maximal abelian extension of $\mathbb{Q}$ is exactly the compositum of all cyclotomic fields $\mathbb{Q}(\zeta_n)$. Moreover, cyclotomic fields are a prime example of **explicit class field theory**, i.e., the explicit construction of the maximal abelian extension. This is part of Kronecker's "*youthful dream*." While general class field theory exists, explicit constructions are still largely open. The case of imaginary quadratic fields is known via complex multiplication, but few other cases are understood; even real quadratic fields remain difficult, though work by Darmon and others has made progress. On the local side, Lubin–Tate theory (via formal groups) provides an explicit construction of the maximal abelian extension, serving as another important gateway to modern topics such as p-adic Galois representations and p-adic Hodge theory.

Topic 2: Weil conjectures for Fermat curves. From the previous topic, we have seen the importance of finite fields.

In this topic, we will use Gauss sums (and Jacobi sums) to prove the Weil conjectures for Fermat curves $x^n + y^n + z^n = 0$ in $\mathbb{P}^2$.

The speaker may follow Ireland–Rosen, Chapters 8 and 11, focusing on the curve case rather than the general exposition.

The zeta function of a variety is the generating series recording the number of points (solutions) over finite fields. For Fermat curves, it can be computed with the help of Gauss sums and Jacobi sums. Using their basic properties together with the Hasse–Davenport relation, one can establish the Weil conjectures in this case. Historically, Weil studied such exponential sums and foresaw the deep geometry behind them, leading him to formulate the Weil conjectures and inaugurated a new era.

To conclude this topic, it would be valuable to give a talk on the geometric intuition behind the Weil conjectures. This task is not easy but highly worthwhile. Again, the speaker should avoid technical details and proofs. The essential insight of Weil is a cohomological picture underlying the zeta function—a surprising idea at the time (imagine geometry over finite fields). The speaker may illustrate classical cohomology for real and complex manifolds; the example of $\mathbb{P}^1$ is sufficient. A toy example of the Lefschetz fixed-point theorem is the circle $S^1 = \mathbb{P}^1_{\mathbb{R}}$ with the inversion map.

Topic 3: Dirichlet density theorem. In our discussion of Gauss sums, we have already encountered the powerful tool of zeta functions and basic representation theory of finite abelian groups, serving as a toy model of harmonic analysis. This topic explores these ideas further.

We will discuss harmonic analysis (Fourier analysis) on the circle S^1, a compact abelian group whose representation theory is analogous to that of finite abelian groups. The basic idea is to examine the spectral side (irreducible characters); this duality can be seen as a toy model for the philosophy of Langlands duality.

Following Serre's book, Chapter VI, we will prove Dirichlet's density theorem: for each n and b coprime to n, there are infinitely many primes congruent to b modulo n. In fact, such primes have positive density, and the theorem is an example of a general density theorem (for cyclotomic fields).

Once again, the speaker may omit technical analytic details (summation techniques) and focus on the key ideas.

Topic 4: Theta series and zeta functions. This topic continues the harmonic analysis theme from Topic 3. We will introduce Poisson summation and use it to prove the functional equation for the one-dimensional theta series, which in turn allows us to determine the Gauss sum completely.

More importantly, following Neukirch, Chapter VII (and Ireland–Rosen, Chapter 16), we will use Poisson summation (together with the Mellin transform) to derive the functional equation for the Riemann zeta function. We will also illustrate the theory of special L-values; the speaker may focus on the zeta function and explain the relation between zeta values and Bernoulli numbers.

Topic 5: p-adic zeta functions. This topic builds on the previous one. We will illustrate the p-adic theory of L-functions. Due to time constraints, we will only touch on the analytic aspects, leaving aside algebraic constructions such as Iwasawa theory.

The main idea for constructing p-adic analytic L-functions involves two steps: first, extract the algebraic part of the special complex L-value (a highly nontrivial task, discussed in Topic 4); second, perform p-adic interpolation (the focus of this topic). We will discuss Kummer congruences for Bernoulli numbers and, using the relation between zeta values and Bernoulli numbers, construct the p-adic zeta function via p-adic interpolation.